Generalized second order NPDEs with similar classes of solutions

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NONLINEARITY, NONLOCALITY AND ULTRAMETRICITY
Belgrade, May 26-30, 2025

Outline

Introduction

- Focus: the largest class of $(1+1)$-second order nonlinear partial differential equations (NPDEs) which can be reduced to an nonlinear ordinary differential equation (NODE) with an imposed form of the wave type.
- Considered NPDEs: the class of reaction-diffusion equations with functional coefficients.
- Generalization: Increased order of nonlinearity for the coefficients so that NPDEs preserve the classes of solutions of the NODEs with wave form.
- Main ingredient: the Lie symmetries of NPDEs, that allows its dimensional reduction to NODEs without changing the classes of solutions (similarity reduction). More precisely, we will use the inverse symmetry and the auxiliary equation methods.
- Examples: Three specific models of nonlinear dynamical systems: a generalized Toda model, the Rossbi model of wave propagation in oceans and generalized forms of the Dodd-Bullough-Mickhailov (DBM) Equation.

Traveling waves solutions

An important class of solutions for NPDEs: traveling wave solutions, emerging when the evolution of the system is expressed in terms of the wave variable $\xi = kx \pm \lambda t$.
To find traveling waves the following procedural steps are fulfilled:

- Reduction of the NPDE to a NODE with the help of the wave variable;
- Finding the solutions of the resulting ODE by an adequate method, as for example by leveraging known solutions from an appropriate auxiliary equation.

For a two-dimensional model defined by the variables x and t, the wave variable is given by $\xi = kx \pm \lambda t$ where λ represents the wave velocity, and k denotes the wavenumber.
The NPDE becomes a NODE, much simpler to solve:

$$F\left(\xi, U, U', U'', ..., U^{(m)}\right) = 0, \ U^{(m)} = \frac{\mathrm{d}^{(m)} U}{\mathrm{d}\xi}. \tag{1}$$

The auxiliary equation technique

The auxiliary equation method involves two essential components:

- Selection of an appropriate auxiliary equation. This step requires choosing a supplementary equation with known solutions.
- Expression of ODE solutions - the solutions of the investigated ODE are then formulated in terms of the auxiliary equation's solutions, usually through a series expansion.

Remarks on the two ingredients:
1) Related to the auxiliary equations: the most commonly employed auxiliary equations are the elliptic Jacobi equation and the Riccati equation, though a variety of other options have also been explored in the literature.
2) Related to the form in which the solutions are sought: several approaches have been proposed for expressing solutions, including the sin-cos method, th-method,Kudryashov method , exponential method, G'/G-method, the functional expansion method, etc.

Wave propagation in oceans, the Rossbi model (I)

The differential equation which describe the *m*-th oscillation mode of the wave's vertical velocity $\phi_m(z)$ in a continuously stratified ocean on the direction z has the form (Rossby):

$$\frac{\mathrm{d}^{(2)}\phi_m(z)}{\mathrm{d}z^2} + \frac{N_m^2(z)}{C_m^2}\phi_m(z) = 0 \qquad (2)$$

One consider for the velocity the boundary conditions:

$$\begin{aligned} \phi_m(z &= -H) = 0 \text{ (ocean floor)} \\ \phi_m(z &= 0) = 0 \text{ (ocean surface)} \end{aligned}$$

Wave propagation in oceans, the Rossbi model (II)

What is known is that, practically, four types of waves appears in this case and have to be found among the solutions of the equation: Kelvin waves, Rossby waves, inertia-gravity waves and mixed Rossby-gravity waves. In the equation (**??**), C_m is a constant and $N_m(z)$ represents the "buoyancy" frequency. Measurements made during El Niño events [**?**] show that the buoyancy frequency $N_m(z)$ has strong variations with the water depth close to the surface and practically vanishes for higher depths. We notice that in the first case (at the surface, $z \in [0, 300]\ m$) one can aproximate $N_m(z)$ with an averaged value around $\overline{N(z)} = 2 \cdot 10^{-4} m \cdot s^{-1}$ So, the equation (**??**) can be linearized in one of the following forms:

$$\ddot{\phi}(z) = 0; z \geq 300 \quad (3)$$

$$\ddot{\phi}(z) + k^2 \phi(z) = 0; z \in [0, 300]; k \equiv \frac{N}{C} = const. \quad (4)$$

A generalized second order NODE

We will come back to the case when, after introducing the wave variable, an ODE is generated and we will consider a large class of second order ODEs of the form:

$$A(u)u'' + B(u)u'^2 + C(u)u' + E(u) = 0 \tag{5}$$

The coefficient functions $A(u)$, $B(u)$, $C(u)$ and $E(u)$ will be considered as polynomials:

$$A(u) = \sum_{\alpha=-n_A}^{n_A} a_\alpha u^\alpha; B(u) = \sum_{\beta=-n_B}^{n_B} b_\beta u^\beta$$

$$C(u) = \sum_{\gamma=-n_C}^{n_C} c_\gamma u^\gamma; E(u) = \sum_{\delta=-n_E}^{n_E} e_\delta u^\delta$$

Specific models of reaction-diffusion equations (I)

Many very interesting and of practical interest equations, with applications in various fields, belong to (**??**). For example:
a) If $C(u) = 0$ we have:

$$A(u)u'' + B(u)u'^2 + E(u) = 0. \tag{6}$$

To this category belongs Dodd-Boulogh-Mikhailov equation, describing fluid flows or QFT systems:

$$-Vuu'' + Vu'^2 + u^3 + 1 = 0 \tag{7}$$

b) If $B(u) = 0, C(u) = 0$ the equation (**??**) becomes:

$$A(u)u'' + E(u) = 0. \tag{8}$$

Specific exemples of equations belonging to this class are:
- Schrodinger equation with cubic nonlinearity:

$$u'' + u^3 - (\beta + \alpha^2)u = 0 \tag{9}$$

Specific models of reaction-diffusion equations (II)

- Nonlinear Klein-Gordon equation

$$k^2(\omega^2 - 1)u'' + \beta u^3 + \alpha u = 0 \tag{10}$$

- Benjamin - Bona - Mahony equation

$$-bPu'' - au^2 - (1 + P)u + k = 0 \tag{11}$$

- Korteweg de Vries equation

$$\delta u'' + \frac{1}{2}u^2 - Vu + k = 0 \tag{12}$$

c) When $B(u) = 0,\ E(u) = 0$ we get:

$$A(u)u'' + C(u)u' = 0$$

It is a Burger type equation, a fundamental partial differential equation occurring in various areas as fluid mechanics, nonlinear acoustics, or gas dynamics. A common specific form is:

$$-Vu' + u\,u' = \nu u'' \tag{13}$$

Here ν is a diffusion coefficient.

d) For $B(u) = 0$ we get:

$$A(u)u'' + C(u)u' + E(u) = 0$$

In particular, it can be the Chafee-Infante (**??**) or Fisher's (**??**) equations:

$$-u'' - Vu' + \lambda(u^3 - u) = 0 \tag{14}$$

$$u'' - u' - u^2 + u = 0 \tag{15}$$

e) If $C(u) = 0,\ E(u) = 0$, we obtain the Hunter-Saxon equation, which in particular has the form:

$$(u - V)u'' + \frac{1}{2}u'^2 = 0 \qquad (16)$$

This equation is important in the theoretical studies of liquid nematic crystals.
f) If $E(u) = 0$ the equation (**??**) becomes:

$$A(u)u'' + B(u)u'^2 + C(u)u' = 0 \qquad (17)$$

In particular, it contains the Buckmaster equation, describing thin viscous fluid sheet flow:

$$4u^3u'' + 12u^2u'^2 + (3u^2 + V)u' = 0 \qquad (18)$$

Let us apply to the generalized reaction-diffusion equation considered before the Lie symmetry analysis. The general expression of the Lie symmetry operator which leaves the reaction diffusion equation invariant can be taken as:

$$U(x,y,t,u) = \frac{\partial}{\partial t} + \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial u} \tag{19}$$

Following the symmetry theory [**?**], the second extension $U^{(2)}$ of (**??**) has to be considered and the invariance condition of the equation (**??**) is given by the relation:

$$\begin{aligned} 0 \quad = \quad & U^{(2)}[u_t - A(x,y,t,u)u_{xy} - B(x,y,t,u)u_x u_y - C(x,y,t,u)u_{2x} - D(\\ & -E(x,y,t,u)u_y - F(x,y,t,u)u_x - G(x,y,t,u)] \end{aligned}$$

The determining system

By equating with zero the coefficient functions of various monomials in derivatives of u, the following partial differential system with11 equations is obtained:

$$
\begin{aligned}
0 &= \xi_u \\
0 &= \eta_u \\
0 &= B\eta_x - D\phi_{2u} \\
0 &= B\xi_y - C\phi_{2u}
\end{aligned}
$$

$$
\begin{aligned}
0 &= A\eta_y - \eta A_y - A_u\phi + A\xi_x - \xi A_x + +2D\xi_y + 2C\eta_x - A_t \\
0 &= A\eta_x + 2D\eta_y - \eta D_y - \xi D_x - D_u\phi - D_t \\
0 &= -A\phi_{2u} + B\xi_x - B\phi_u + B\eta_y - B_t - B_x\xi - B_u\phi - B_y\eta
\end{aligned}
$$

$$
\begin{aligned}
0 = \; & -\eta_t + F\eta_x - B\phi_x + E\eta_y - E_t - E_x\xi - E_y\eta - E_u\phi \\
& + A\eta_{xy} - A\phi_{xu} + C\eta_{2x} + D\eta_{2y} - 2D\phi_{yu}
\end{aligned} \qquad (21)
$$

$$
0 = -\xi_t - B\phi_y + F\xi_x + E\xi_y - F_t - F_x\xi - F_y\eta - F_u\phi
$$

The inverse symmetry problem

Two approaches are now possible: (i) DIRECT PROBLEM: to find the symmetries of a given evolutionary equation, which means to choose concrete forms for $A(x, y, t, u)$, $B(x, y, t, u)$, $C(x, y, t, u)$, $D(x, y, t, u)$, $E(x, y, t, u)$.
(ii) INVERSE PROBLEM: to take as unknown variables $A(x, y, t, u)$, $B(x, y, t, u)$, $C(x, y, t, u)$, $D(x, y, t, u)$, $E(x, y, t, u)$, $F(x, y, t, u)$, $G(x, y, t, u)$ and imposing a concrete form of the symmetry group. The second approach allows us to determine all the equations which are equivalent from the point of view of the symmetry group they do admit.

The functional expansion for polynomial coefficients

In most cases, the limit of the expansion in equation (**??**) is determined by employing a balancing procedure between the highest order of the derivative and the highest order of the nonlinearity in the ODE [**?**]. Considering that, for nearly all equations of physical interest, the coefficient functions $A(U)$, $B(U)$, and $C(U)$ are polynomials of degree at most 2:

$$A(U) = A_0 + A_1 U + A_2 U^2,\ B(U) = B_0 + B_1 U + B_2 U^2,\ C(U) = C_0 + C_1 U + C_2 U^2 \tag{22}$$

we choose for the free nonlinear term $E(U)$ of order at most 5:

$$E(U) = E_0 + E_1 U + E_2 U^2 + E_3 U^3 + E_4 U^4 + E_5 U^5. \tag{23}$$

To enhance the generalization of the results, we have opted for a series expansion of order 3 in the development of the solution, leading us to:

$$U(\xi) = P_0 + P_1 G(\xi) + P_2 G(\xi)^2 + P_3 G(\xi)^3. \tag{24}$$

Compatibility with the elliptic equation

Compatibility is evidenced by the identical solutions shared between both equations, or by the fact that the solution of the NODE (**??**) can be expressed in terms of the solution to the auxiliary equation. Sequentially substituting the relations (**??**)-(**??**) into the general equation (**??**) we will obtain relationships among the coefficients P_i, $i = \overline{0,3}$, g_j, $j = \overline{0,4}$, (A_k, B_k, C_k), $k = \overline{0,3}$, E_n, $n = \overline{0,5}$ which together will yield the most general form of nonlinear equations solvable using the auxiliary equation (**??**).
To achieve this, we methodically group the coefficients based on the powers of G, thereby deriving an algebraic system.
When considering the specific case where $l = 2$, the auxiliary equation becomes the elliptic Jacobi equation and the derived system of equations is further constrained by the coefficients of G'.
These additional constraints refine the algebraic system, ensuring that the nonlinear equation is consistent with the auxiliary equation framework.

Constraint system solutions

The procedure presented before admits two evident solutions:
i) In the first case, the coefficients C_0, C_1, C_2, P_0 are arbitrary, while $P_1 = P_2 = P_3 = 0$. These values suggest that the solution corresponds to a constant function.
ii) In the second case, the coefficients P_0, P_1, P_2, P_3 are arbitrary, whereas $C_0 = C_1 = C_2 = 0$. This outcome is more suitable as it aligns with a particular class of equations where the function $C(U) = 0$.
By selecting the second solution and substituting it into the system (**??**), we proceed to group the coefficients according to the powers of G. This grouping reveals that the highest order in the resulting expression is 15, leading to the formulation of a new algebraic system comprised of 16 equations with 21 unknowns. The resulting system yields 18 potential solutions, as determined through symbolic computation using Maple.

Solutions of the determining system. Case I

We will present only a subset of these solutions, specifically those deemed most relevant to our case. The derived values provide the most comprehensive form of the general NODE, which is given by:

$\left(A_0 + A_1 U + A_2 U^2\right) U'' + \left(B_0 + B_1 U + B_2 U^2\right) U'^2 + E_0 + E_1 U + E_2 U^2 + E_3 U^3 + E_4 U^4 + E_5 U^5 = 0,$

compatible with the elliptic equation as an auxiliary equation:

$$\left(G'\right)^2 = g_0 + g_1 G(\xi) + g_2 G(\xi)^2 + g_3 G(\xi)^3 ,$$

(25)

having the solution form: $U(\xi) = P_0 + P_1 G(\xi)$.

Solutions of the determining system. Case II

Other choices provide the general form of the equation:
$\left(A_0 + A_1 U - 1/2\, B_1 U^2\right) U'' + (B_0 + B_1 U)\, U'^2 + E_0 + E_1 U + E_2 U^2 + E_3 U^3 + E_4 U^4 = 0,$
It is compatible with the auxiliary equation:
$(G')^2 = g_0 + g_1 G(\xi) + g_2 G(\xi)^2 + g_3 G(\xi)^3 + g_4 G(\xi)^4,$
The solution form is in this case: $U(\xi) = P_0 + P_1 G(\xi)$.

Solutions of the determining system. Case III

Another possible form of generalized equation is:

$$\left(A_0 + A_1 U + A_2 U^2\right) U" + \left(B_0 + B_1 U + B_2 U^2\right) U'^2 + E_0 + E_1 U + E_2 U^2 + E_3 U^3 + E_4 U^4 + E_5 U^5 = 0,$$

It is compatible with the auxiliary equation:

$$(G')^2 = g_0 + g_1 G(\xi) + g_2 G(\xi)^2 + g_3 G(\xi)^3 + g_4 G(\xi)^4,$$

The solution is in this case of the following form:

$$U(\xi) = P_0 + P_1 G(\xi) + P_2 G(\xi)^2 .$$

We will consider now as another example the Dodd–Bullough–Mikhailov Equation. It falls within the category of the general equation of reaction-diffusion and it is significant in the study of nonlinear systems, particularly within the fields of hydrodynamics and QFT.

$$u_{xt}u - u_x u_t + u^3 + 1 = 0. \tag{26}$$

The ODE correspondig to the DBM equation has the form:

$$-\lambda UU'' + \lambda U'^2 + U^3 + 1 = 0. \tag{27}$$

To advance our understanding, it is essential to investigate the equivalence of the equation in question with the previously discussed cases.
The equation aligns with the previously discussed examples,more precisely, example I, under the following conditions:
$A_0 = A_2 = B_1 = B_2 = 0, A_1 = -\lambda, B_0 = \lambda, E_5 = E_4 = E_2 = E_1 = 0$ and E[illegible] $= E$[illegible] $= 1$

The Toda field model (I)

- A Toda field theory is a nice example which can appear in mathematics and physics, specifically in the study of field theory and partial differential equations.
- The theory can be specified by a Lie algebra and usually refer to theories with a finite Lie algebra. Depending on the rank of its associated Lie algebra, the Toda field ϕ represents in fact a collection of scalar fields $\{\phi^i, i = 1, ..., r\}$, with r the rank.

The Toda field model (II)

- In a $2D$-Minkowski spacetime, x, t, the Toda Lagrangian can be written:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{\beta^2} \sum_{i=1}^{r} n_i \exp \beta(\alpha_i \phi^i) \tag{28}$$

  The integers n_i are Kac or Dynkin labels, m is the mass and β the coupling constant,
  α_i is the i-th simple root, and provides a basis for the Cartan subalgebra on $\mathcal{R}^r$.
- At the classical level, the Toda field theories represent nonlinear integrable models and their solutions describe solitons.
- At the quantum level, by liniarizing the Toda fields we can generate quantum models with the spectrum consisting of one or many massive particles.
- In Cosmology, the Toda fields could describe tachionic fields appearing in inflation models.

Conclusion

- The primary focus has been on identifying and analyzing a generalized class of equations that maintain consistency with the solutions of the general elliptic equation used as auxiliary equation.
- The importance of the relationships between the coefficients of the equations under consideration demonstrates that these relationships are important in determining whether solutions of the Jacobi equation can be generalized to other nonlinear systems.
- The investigation was centered on the Dodd-Boulogh-Mikhailov equation, with particular attention given to the role of derivative-free terms.
- This extension not only broadens the applicability of the of Jacobi equation but also highlights the potential for the solutions to be applied within a wider range of nonlinear differential systems

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Thank you for your attention!