Primordial Black Holes in Induced Gravity

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Induced gravity models and the corresponding CCMs

- The induced gravity model with two scalar fields:

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\xi}{2}\sigma^2\tilde{R} - \tfrac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \tilde{V}(\sigma,\chi)\right], \qquad (1)$$

where ξ is a positive constant, the potential $\tilde{V}$ is a differentiable function, $M_{\rm Pl}$ is the Planck mass.

- The conformal transformation of the metric $g_{\mu\nu} = \frac{\xi\sigma^2}{M_{\rm Pl}^2}\tilde{g}_{\mu\nu}$.
- The action of the two-field CCM:

$$S_E = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{y}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V_E\right], \qquad (2)$$

where

$$\phi = M_{\rm Pl}\sqrt{6+\frac{1}{\xi}}\,\ln\left(\frac{\sigma}{M_{\rm Pl}}\right), \quad y = \frac{M_{\rm Pl}^2}{\xi\sigma^2} = \frac{1}{\xi}\exp\left(-2\sqrt{\frac{\xi}{6\xi+1}}\frac{\phi}{M_{\rm Pl}}\right), \qquad (3)$$

and the potential $V_E = y^2(\phi)\tilde{V}(\sigma(\phi),\chi)$.

Evolution equations in the Einstein frame

• In the spatially flat Friedman-Lemaitre-Robertson-Walker (FLRW) universe with the interval

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right),$$

• The Hubble parameter $H(t)$ is the logarithmic derivative of the scale factor: $H = \dot{a}/a$.

• It is suitable to consider the e-folding number $N = \ln(a/a_e)$, where a_e is a constant, as an independent variable during inflation. We use the relation $\frac{d}{dt} = H\,\frac{d}{dN}$.

• The evolution equations have the following form:

$$H^2 = \frac{2V_E}{6M_{\rm Pl}^2 - {\phi'}^2 - y{\chi'}^2}\,, H' = -\frac{H}{2M_{\rm Pl}^2}\left[{\phi'}^2 + y{\chi'}^2\right], \qquad (4)$$

where primes denote derivatives with respect to N.

• The field equations system:

$$\phi'' = (\epsilon_1 - 3)\phi' + \frac{1}{2}\,\frac{dy}{d\phi}\,{\chi'}^2 - \frac{6M_{\rm Pl}^2 - y{\chi'}^2 - {\phi'}^2}{2V_E}\,\frac{\partial V_E}{\partial \phi}\,,$$

$$\chi'' = (\epsilon_1 - 3)\chi' - \frac{1}{y}\,\frac{dy}{d\phi}\,\chi'\phi' - \frac{6M_{\rm Pl}^2 - {\phi'}^2 - y{\chi'}^2}{2yV_E}\,\frac{\partial V_E}{\partial \chi}\,.$$

Parameters

- The slow-roll parameters ϵ_1 and ϵ_2 are

$$\epsilon_1 = -\frac{H'}{H} = \frac{1}{2M_{\rm Pl}^2}\left[{\phi'}^2 + y{\chi'}^2\right], \quad \epsilon_2 = \frac{\epsilon_1'}{\epsilon_1}. \tag{5}$$

- The inflationary parameters: the spectral index and the tensor-to-scalar ratio

$$n_s = 1 - 2\epsilon_1 - \epsilon_2, \qquad r \approx 16\epsilon_1. \tag{6}$$

- The amplitude of scalar perturbations

$$A_s = \frac{2H^2}{\pi^2 M_{\rm Pl}^2 r}. \tag{7}$$

• The slow-roll parameter $\epsilon_1 < 1$ during inflation. In the slow-roll regime, $|\epsilon_2| < 1$ as well. The slow-roll regime is violated and the first stage of inflation ends, sufficiently large peaks appear in the power spectrum of scalar perturbations, which lead to the possible PBHs after inflation.

• We estimate the value of N_* by the relation:

$$2\epsilon_1(N_*) - \frac{\epsilon_2(N_*)}{2} \simeq 3 \tag{8}$$

• The second stage of inflation leads to the possible generation of PBH begins at the point N_*. To get the duration of inflation N_{tot} we use expression $\epsilon_1(N_{tot}) = 1$.

• The formula for calculation of PBH mass

$$M_{PBH} \approx \frac{M_{\rm Pl}^2}{H(N_{tot})} \exp\left(2(N_{tot} - N_*)\right) . \tag{9}$$

Inflationary model

We consider the following potential:

$$\tilde{V}(\sigma,\chi) = \lambda\sigma^4 \left(F_1(\chi) + F_2(\chi)\mathrm{e}^{\gamma\,[\ln(\sigma/M_{\rm Pl})]^{2\alpha}}\right), \qquad (10)$$

where

$$F_1(\chi) = \left(1 - \frac{\chi^2}{\chi_0^2}\right)^2 - d\,\frac{\chi}{\chi_0}, \qquad F_2(\chi) = \frac{c_2\chi^2}{\chi_0^2} + c_0\,, \qquad (11)$$

α, γ, λ, χ_0, c_0, c_2 and d are constants. Note that the potential $\tilde{V}$ is real even if $\ln(\sigma/M_{\rm Pl}) < 0$ and α is not an integer number.
In the Einstein frame, we get

$$V_E(\phi,\chi) = V_0\left(F_1(\chi) + F_2(\chi)\mathrm{e}^{\beta\left(\frac{\phi^2}{M_{\rm Pl}^2}\right)^{\alpha}}\right), \qquad (12)$$

where

$$V_0 = \frac{\lambda M_{\rm Pl}^4}{\xi^2}, \qquad \beta = \gamma\left(\frac{\xi}{1+6\,\xi}\right)^{\alpha}. \qquad (13)$$

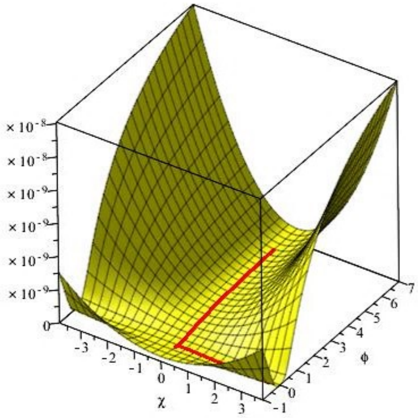

Figure: The potential $V(\phi, \xi)$ and the trajectory

Numerical solutions of the evolution equations

We solve system solutions of the evolution equations numerically to analyze the evolution of scalar fields during inflation and to get values of inflationary parameters. We define the e-folding number N in such a way that $N = 0$ corresponds to the moment at which inflationary parameters are calculated. For the values of models parameters

$$\begin{aligned} V_0 &= 10^{-10} M_{\rm Pl}^4, \quad \alpha = -0.37, \quad \beta = -1.8, \\ \chi_0 &= 3.5\, M_{\rm Pl}, \quad c_0 = 12, \quad c_2 = 147, \quad d = 10^{-3}, \end{aligned} \tag{14}$$

we get inflationary parameters

$$n_s = 0.9622, \quad r = 0.0266, \quad A_s = 2.10 \cdot 10^{-9}\,, \tag{15}$$

α	β	$\phi_0/M_{\rm Pl}$	n_s	r	N_*	N_{tot}
-0.40	-2	5.867	0.962	0.027	38.2	61.9
-0.40	-1.8	5.502	0.959	0.027	35.4	58.9
-0.40	-1.5	4.936	0.954	0.027	31.1	53.5
-0.37	-2	6.008	0.965	0.026	40.8	64.2
-0.37	-1.8	5.623	0.962	0.026	37.6	60.7
-0.37	-1.5	5.018	0.957	0.028	32.8	54.9
-0.35	-2	6.104	0.967	0.025	42.7	65.4
-0.35	-1.8	5.701	0.964	0.026	39.2	61.9
-0.35	-1.5	5.072	0.959	0.027	34.1	56.2

Table: Dependence of inflation parameters, duration of the first stage of inflation N_* and total duration of inflation N_{tot} on the model parameters α and β. Other model parameters are chosen as follows:

$$V_0 = 10^{-10} M_{\rm Pl}^4, \quad \chi_0 = 3.5\, M_{\rm Pl}, \quad c_0 = 12, \quad c_2 = 147, \quad d = 0.003.$$

d	N_{tot}	$N_{tot} - N_*$	$M_{PBH}/M_{\rm Pl}$	$M_{PBH}/M_\odot$	M_{PBH}/g
0.001	64.5	26.9	$8.57 \cdot 10^{28}$	$1.87 \cdot 10^{-10}$	$3.72 \cdot 10^{23}$
0.002	62.1	24.5	$7.07 \cdot 10^{26}$	$1.54 \cdot 10^{-12}$	$3.07 \cdot 10^{21}$
0.003	60.7	23.1	$4.31 \cdot 10^{25}$	$9.40 \cdot 10^{-14}$	$1.87 \cdot 10^{20}$
0.007	57.7	20.1	$1.18 \cdot 10^{23}$	$2.57 \cdot 10^{-16}$	$5.14 \cdot 10^{17}$
0.01	56.5	18.9	$9.92 \cdot 10^{21}$	$2.16 \cdot 10^{-17}$	$4.30 \cdot 10^{16}$

Table: The dependence of duration of inflation N_{tot} and the PBH mass M_{PBH} from the model parameter d. Other model parameters are given by (14). The end of the first stage of inflation is at $N_* = 37.6$ independent on d.

The current estimation of the mass region of PBHs considered as candidates for dark matter is $10^{-17}\,M_\odot \leqslant M_{PBH} \leqslant 10^{-12} M_\odot$, where $M_\odot$ is the Solar mass (Ozsoy (2023) and references therein). As shown in Table 2, the proposed model with $0.002 \leqslant d \leqslant 0.01$ allows us to reproduce the masses of the PBH from this interval.

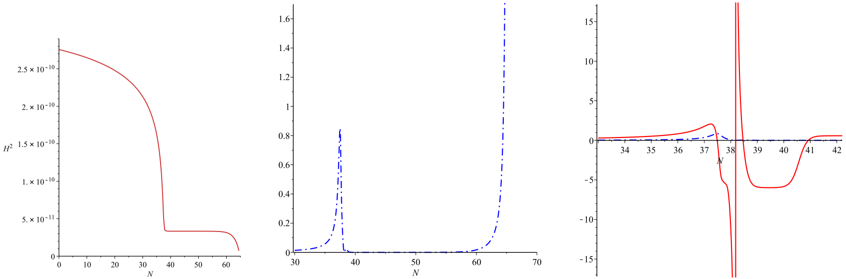

Figure: The behaviour of $H^2(N)$ (left picture). The behavior of slow-roll parameters ϵ_1 (blue dash-dot curve) and ϵ_2 (red solid curve) during inflation.

Evolution equations

- Nonminimally coupled model can be described by the following action:

$$S = \int d^4x\sqrt{-g}\left[U(\sigma)R - \tfrac{c}{2}g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu} + V(\sigma)\right], \tag{16}$$

where $U(\sigma)$ and $V(\sigma)$ are differentiable functions of the scalar field σ, g is the determinant of the metric tensor $g_{\mu\nu}$.

- The Friedman equations follows:

$$6UH^2 + 6\dot{U}H = \tfrac{c}{2}\dot{\sigma}^2 + V, \tag{17}$$

$$2U\left(2\dot{H} + 3H^2\right) + 4\dot{U}H + 2\ddot{U} + \tfrac{c}{2}\dot{\sigma}^2 - V = 0, \tag{18}$$

and the variation of ((16)) with respect to σ gives the Klein–Gordon equation,

$$c\left(\ddot{\sigma} + 3H\dot{\sigma}\right) + V_{,\sigma} = 6\left(\dot{H} + 2H^2\right)U_{,\sigma} \tag{19}$$

- Combining Eqs. (17) and (18), one obtains

$$4U\dot{H} - 2HU_{,\sigma}\dot{\sigma} + 2\left(U_{,\sigma\sigma}\dot{\sigma}^2 + U_{,\sigma}\ddot{\sigma}\right) + c\dot{\sigma}^2 = 0. \tag{20}$$

Superpotential method for models with nonminimal coupling

• Following to A. Y. Kamenshchik, A. Tronconi, G. Venturi and S. Y. Vernov, Phys. Rev. D **87** (2013) no.6, 063503 doi:10.1103/PhysRevD.87.063503 [arXiv:1211.6272 [gr-qc]] we introduce the function $G(\sigma)$ defined as

$$\dot{\sigma} = G(\sigma) \tag{21}$$

• The potential $V(\sigma)$ can be easily obtained by inverting the Friedmann equation (17):

$$V(\sigma) = 6UH^2 + 6U_{,\sigma}GH - \frac{G^2}{2} = 6H^2U\left(1 + \frac{3\,U_{,\sigma}^2}{U}\right) - \frac{(G - 6U_{,\sigma}H)^2}{2}. \tag{22}$$

where $\sigma = \sigma(t)$ is monotonic.

- G and $H(\sigma)$ are related as

$$H(\sigma) = -\left[\int^{\sigma} \frac{2G_{,\tilde{\sigma}}U_{,\tilde{\sigma}} + (2U_{,\tilde{\sigma}\tilde{\sigma}} + 1)G}{4U^{3/2}}\, d\tilde{\sigma} + c_0\right]\sqrt{U(\sigma)}, \tag{23}$$

or

$$G(\sigma) = \left[\int^{\sigma} \frac{U_{,\tilde{\sigma}}H - 2UH_{,\tilde{\sigma}}}{U_{,\tilde{\sigma}}} e^{\Upsilon}\, d\tilde{\sigma} + \tilde{c}_0\right] e^{-\Upsilon(\sigma)}, \tag{24}$$

where

$$\Upsilon(\sigma) \equiv \tfrac{1}{2}\int^{\sigma} \frac{2U_{,\tilde{\sigma}\tilde{\sigma}} + 1}{U_{,\tilde{\sigma}}}\, d\tilde{\sigma}$$

and where c_0 is an integration constant

Minimally coupled model

- Let us first consider the minimally coupled model with $U = U_0 = M^2/2$:

$$\dot{H} = -\frac{\dot{\phi}^2}{2M^2}, \quad G = -2M^2 H_{,\phi}. \tag{25}$$

- The potential

$$V(\phi) = 3M^2 H^2 - 2M^4 H_{,\phi}^2. \tag{26}$$

We are interested in solutions with the scalar field and the Hubble parameter asymptotically approaching some (de Sitter) fixed point: $\phi \to \phi_0$, $H \to H_0$. These solutions are realized if

$$\lim_{\phi\to\phi_0} \dot{\phi} = 0, \quad \Rightarrow \quad \lim_{\phi\to\phi_0} H_{,\phi} \to 0. \tag{27}$$

If $H(\phi)$ is regular in $\phi_0 \neq 0$, then its Taylor expansion around ϕ_0 has the following form:

$$H(\phi) = \sum_{n=0}^{\infty} h_n \left(\frac{\phi}{\phi_0} - 1\right)^n \equiv \sum_{n=0}^{\infty} h_n \left(\frac{\delta\phi}{\phi_0}\right)^n , \tag{28}$$

and the condition (27) is satisfied if $h_1 = 0$.

Stability

The stability of the de Sitter solution ($\phi = \phi_0$) with respect to homogeneous and isotropic perturbations. On expanding the homogeneous Klein–Gordon equation around the scalar field fixed point ϕ_0, to linear order one finds

$$3\,M^2 V_{,\phi}(\phi_0) + V(\phi_0)\delta\phi_{,NN} + 3V(\phi_0)\delta\phi_{,N} + 3\,M^2 V_{,\phi\phi}(\phi_0)\delta\phi = 0 \qquad (29)$$

where the first term is zero, $\delta\phi = \phi - \phi_0$, $V(\phi_0) = 3\,M^2 h_0^2$,

$$V_{,\phi\phi}(\phi_0) = 12\frac{M^2}{\phi_0^2}h_0 h_2\left(1 - \frac{4}{3}\frac{h_2}{h_0}\frac{M^2}{\phi_0^2}\right) = \frac{V(\phi_0)}{M^2}\left(\gamma - \frac{\gamma^2}{3}\right). \qquad (30)$$

and h_0 and h_2 are the Taylor coefficients of the expansion of H around ϕ_0.

This invariance is simply related to the fact that for any given potential, on linearizing the KG equation close to ϕ_0, one finds two independent solutions. Indeed, the linearized equation for $\delta\phi$ is easily solved by

$$\delta\phi = c_1\, e^{-N(3-\gamma)} + c_2\, e^{-\gamma N}. \qquad (31)$$

The solution (31) is stable if

$$0 < \gamma < 3.$$

Nonminimally coupled inflaton

- On may define an effective potential [1]

$$V_{eff}(\sigma) = \frac{U_0^2 V(\sigma)}{U^2(\sigma)}, \tag{32}$$

the new, dimensionless, function

$$A = \frac{U_0^2}{U^2}\left(1 + \frac{3U_{,\sigma}^2}{U}\right) \tag{33}$$

and

$$Y = \sqrt{\frac{U_0}{U}}\left(H + \dot{\sigma}\frac{U_{,\sigma}}{2U}\right). \tag{34}$$

In terms of these functions, Eqs. (17) and (20) take the following form:

$$6U_0 Y^2 = \frac{A}{2}\dot{\sigma}^2 + V_{eff}, \tag{35}$$

which has the same structure as the Friedmann equation (17) for a minimally coupled inflaton.

[1] E. O. Pozdeeva, M. A. Skugoreva, A. V. Toporensky and S. Y. Vernov, JCAP **12** (2016), 006 [arXiv:1608.08214 [gr-qc]];M. A. Skugoreva, A. V. Toporensky and S. Y. Vernov, Phys. Rev. D **90** (2014) no.6, 064044 [arXiv:1404.6226 [gr-qc]]

The corresponding acceleration equation is

$$\dot{Y} = -\frac{A\sqrt{U}}{4U_0^{3/2}}\,\dot{\sigma}^2 \Leftrightarrow Y_{,\sigma} = -\frac{A\sqrt{U}}{4U_0^{3/2}}\,G \qquad (36)$$

with $G = \dot{\sigma}$. Furthermore, the Klein–Gordon equation (19) takes the following form:

$$\ddot{\sigma} = -3\sqrt{\frac{U}{U_0}}Y\dot{\sigma} - \frac{A_{,\sigma}}{2A}\dot{\sigma}^2 - \frac{V_{eff\,,\sigma}}{A}. \qquad (37)$$

Therefore, the condition $V_{eff\,,\sigma} = 0$ clearly corresponds to de Sitter solutions. Such solutions are stable for $V_{eff\,,\sigma\sigma} > 0$ and unstable for $V_{eff\,,\sigma\sigma} < 0$, provided the condition $U(\sigma) > 0$ is satisfied. Let us note that any nonminimally coupled model can be transformed to the corresponding EF model through a conformal transformation of the metric and a field redefinition. In the EF, the effective potential V_{eff} is equal to the potential V and $Y(\sigma)$ is the Hubble parameter.

• Once the functional dependence of $U(\sigma)$ and $Y(\sigma)$ is properly chosen, one easily gets the corresponding effective potential

$$V_{eff}(\sigma) = 2U_0\left(3Y^2 - \frac{4U_0^2}{A\,U}Y_{,\sigma}^2\right) \tag{38}$$

and the associated inflaton potential

$$V(\sigma) = 2\frac{U^2}{U_0}\left(3Y^2 - \frac{4U_0^2}{A\,U}Y_{,\sigma}^2\right) = 6\frac{U^2}{U_0}Y^2 - \frac{8U^4}{U_0\left(3U_{,\sigma}^2 + U\right)}Y_{,\sigma}^2. \tag{39}$$

• Finally, one finds from Eqs. (34) and (36) that

$$H = \sqrt{\frac{U}{U_0}}\left(Y + \frac{2U\,U_{,\sigma}Y_{,\sigma}}{U + 3U_{,\sigma}^2}\right), \quad G = -\frac{4U^{5/2}Y_{,\sigma}}{\sqrt{U_0}\left(U + 3U_{,\sigma}^2\right)} \tag{40}$$

• Thus, on following the same procedure as used for the minimally coupled case, one can find, for any given choice of $U(\sigma)$, the conditions for Y leading to specific inflationary evolutions in the Jordan frame.

Induced gravity models

- The induced gravity (IG) case with

$$U = \frac{\xi\sigma^2}{2}. \tag{41}$$

From Eq. (40), one has

$$G = -\alpha\sigma^3 Y_{,\sigma}, \qquad H = \beta\sigma Y + \alpha\sigma^2 Y_{,\sigma}\,, \tag{42}$$

with $\beta \equiv \sqrt{\xi}/M$ and $\alpha \equiv 2\xi\beta/(1+6\xi)$. One can easily write the corresponding potential[2]

$$V = \frac{\xi^2\sigma^4\left(3(1+6\xi)Y^2 - 2\xi\sigma^2 Y_{,\sigma}^2\right)}{(1+6\xi)\,M^2}, \tag{43}$$

[2] Using $\sigma = \sigma_0 \exp\left(\frac{\xi\phi}{\sqrt{2U_0(\xi+6\xi^2)}}\right)$ and Eq. (32), one can calculate the potential in the EF in terms of ϕ.

If the inflaton evolves to some fixed point σ_0 (without loss of generality we consider $\sigma_0 > 0$) and we Taylor expand Y around it

$$Y = \sum_{n=0}^{\infty} y_n \left(\tfrac{\sigma}{\sigma_0} - 1\right)^n . \tag{44}$$

A very simple potential possibly leading to an amplification can be obtained by choosing $Y = y_0 + y_2 \left(\sigma/\sigma_0 - 1\right)^2$. In such a case,

$$\sigma Y_{,\sigma} = 2y_2 \frac{\sigma}{\sigma_0} \left(\frac{\sigma}{\sigma_0} - 1\right) \tag{45}$$

and the potential is

$$V = \frac{\xi^2 \sigma^4 y_2^2 \left(3(1+6\xi)\left(\frac{y_0}{y_2} + \left(\frac{\sigma}{\sigma_0} - 1\right)^2\right)^2 - 8\xi \frac{\sigma^2}{\sigma_0^2}\left(\frac{\sigma}{\sigma_0} - 1\right)^2\right)}{M^2(1+6\xi)}. \tag{46}$$

The (general) behavior close to σ_0 , where $Y \sim y_0 + y_2 \left(\sigma/\sigma_0 - 1\right)^2$. One has

$$\dot{\sigma} \simeq -2y_2\alpha\sigma_0^2 \left(\frac{\sigma}{\sigma_0} - 1\right), H \simeq \beta\sigma_0 y_0 \left(1 + \left(1 + 2\frac{\alpha y_2}{\beta y_0}\right)\left(\frac{\sigma}{\sigma_0} - 1\right)\right) , \qquad (47)$$

where we observe that H now approaches the constant value linearly on varying the field σ.

- Conclusion: 1) PBH; 2)Stability of USL; 3)Induced gravity.

Thank your for attention.

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