NONLINEARITY, NONLOCALITY AND ULTRAMETRICITY

Branko Dragovich's 80th Birthday

Representations of the p-adic rotation group: towards p-adic qubits and quantum computing

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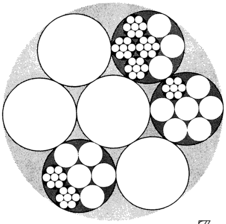
Outline

p-Adic numbers

- Field $\mathbb{Q}_p \ni x = \sum_{n \geq n_0} x_n p^n$, with $n_0 \in \mathbb{Z}$, $x_n \in \{0, 1, \ldots, p-1\}$
- Ring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \text{ s.t. } n_0 \geq 0\}$, inverse limit $\mathbb{Z}_p \simeq \varprojlim \{\mathbb{Z}/p^k\mathbb{Z}\}_{\mathbb{N}}$
- $\mathbb{Q}_p$ is totally disconnected

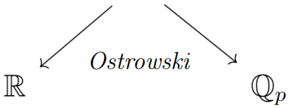


$\mathbb{Z}_7$

- The algebraic closure of $\mathbb{Q}_p$ has infinite degree

Motivations

- In physical observations, we deal with $\mathbb{Q}$



- p-Adic quantum mechanics [1]
  1. p-adic configuration space
  2. symmetry group
  3. projective unitary irreducible representations (irreps)

> Symmetry group of rotations on $\mathbb{Q}_p^3$
> Irreps for p-adic angular momentum and spin
> 2-dimensional irreps $\leftrightarrow$ p-adic qubit
> p-adically controlled quantum logic gates

[1] Vladimirov, Volovich, Zelenov, *p-Adic Analysis and Mathematical Physics*, 1994.

Quadratic forms on $\mathbb{Q}_p^3$

- $Q(x,y,z) = ax^2 + by^2 + cz^2 \doteq \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$
- $\mathrm{SO}(Q) = \{L \in \mathrm{M}(3,\mathbb{Q}_p) \text{ s.t. } L^\top Q L = Q,\ \det L = 1\}$
- $Q \sim Q' \implies \mathrm{SO}(Q) \simeq \mathrm{SO}(Q')$

2 classes of quadratic forms on $\mathbb{Q}_p^3$: (likewise $\mathbb{R}$)

Q_0 indefinite	Q_+ definite
$\Downarrow$	$\Downarrow$
$\mathrm{SO}(Q_0)$ not compact	$\mathrm{SO}(Q_+)$ compact

Special orthogonal group on $\mathbb{Q}_p^3$

Unique definite quadratic form on $\mathbb{Q}_p^3$:

$$Q_+(\boldsymbol{x}) = x^2 - vy^2 + pz^2 \doteq \mathrm{diag}(1, -v, p)$$

where v is a non-square p-adic unit

Unique compact group
$\mathrm{SO}(3)_p := \mathrm{SO}(Q_+)$

Unique definite quadratic form on $\mathbb{Q}_p^4$: $Q_+^{(4)}(\boldsymbol{x}) = x^2 - vy^2 + pz^2 - pvt^2$

No definite quadratic forms on $\mathbb{Q}_p^n$ for $n \geq 5$

Basic facts about $\mathrm{SO}(3)_p$ [2]

- $\mathrm{SO}(3)_p \subset \mathrm{M}(3, \mathbb{Z}_p)$ is compact
- The elements of $\mathrm{SO}(3)_p$ are rotations around a fixed axis $\mathbb{Q}_p \boldsymbol{n}$ of $\mathbb{Q}_p^3$
- The x-axis cannot be rotated to the z-axis
- There are three groups $\mathrm{SO}(2)_{p,\partial}$, parametrised as C_∂ or $P^1(\mathbb{Q}_p)$

W.r.t. an orthogonal basis $(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{n})$ of $\mathbb{Q}_p^3$,

$$\mathrm{SO}(3)_p \ni \mathcal{R}_{\boldsymbol{n}}(\sigma) = \begin{pmatrix} \frac{1-\delta\sigma^2}{1+\delta\sigma^2} & -\frac{2\delta\sigma}{1+\delta\sigma^2} & 0 \\ \frac{2\sigma}{1+\delta\sigma^2} & \frac{1-\delta\sigma^2}{1+\delta\sigma^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{array}{l} \sigma \in \mathbb{Q}_p \cup \{\infty\} \\ \delta = Q_+(\boldsymbol{w})/Q_+(\boldsymbol{v}) \end{array}$$

[2] Di Martino, Mancini, Pigliapochi, Svampa, Winter, "Geometry of the p-Adic Special Orthogonal Group $\mathrm{SO}(3)_p$", *Lobachevskii J. Math.* **44**(6), pp. 2135-2159 (2023).

Euler and nautical angles

Any $R \in \mathrm{SO}(3)_{\mathbb{R}}$ can be written as any of the compositions

$$R_x R_y R_z, \quad R_y R_z R_x, \quad R_z R_x R_y, \quad R_x R_z R_y, \quad R_z R_y R_x, \quad R_y R_x R_z$$
$$R_x R_y R_x, \quad R_x R_z R_x, \quad R_y R_x R_y, \quad R_y R_z R_y, \quad R_z R_x R_z, \quad R_z R_y R_z,$$

respectively of certain angles $\theta, \psi, \phi \in \mathbb{R}$

Theorem

Any $\mathcal{R} \in \mathrm{SO}(3)_p$, can be written as any of the compositions

$$\mathcal{R}_z \mathcal{R}_y \mathcal{R}_x, \quad \mathcal{R}_z \mathcal{R}_x \mathcal{R}_y, \quad \mathcal{R}_x \mathcal{R}_y \mathcal{R}_z, \quad \mathcal{R}_y \mathcal{R}_x \mathcal{R}_z,$$

respectively of certain parameters $\sigma, \tau, \omega \in \mathbb{Q}_p \cup \{\infty\}$

None of the other decompositions exist.

Each is exactly twofold, and unique if we restrict the parameters:

$$\mathcal{R} = \mathcal{R}_z(\omega)\mathcal{R}_y(\tau)\mathcal{R}_x(\sigma) = \mathcal{R}_z(\infty)\mathcal{R}_z(\omega)\mathcal{R}_y(\infty)\mathcal{R}_y(-\tau)\mathcal{R}_x(\infty)\mathcal{R}_x(\sigma)$$

Programme

To classify all projective irreps

$$U\colon \mathrm{SO}(3)_p \to \mathrm{PU}(\mathbb{C}^n) := \mathrm{U}(n)/\mathrm{U}(1).$$

There exists a unique Haar measure μ on the compact group $\mathrm{SO}(3)_p$

- Regular representation of $\mathrm{SO}(3)_p$ on $\mathrm{L}^2(\mathrm{SO}(3)_p, \mu)$

Peter-Weyl Theorem

The regular representation of $\mathrm{SO}(3)_p$ is decomposable into a direct sum of all the irreps of $\mathrm{SO}(3)_p$

- Orthogonality relations between projective irreps

1st approach

Haar measure on p-adic Lie groups [3]

Theorem

Let G be a p-adic Lie group, and let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ be an atlas for G. If μ is the left Haar measure on G, then, for every Borel set $E \in \mathcal{B}_G$, and every $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$,

$$\mu(E \cap U_\alpha) = \int_{\varphi_\alpha(E \cap U_\alpha)} \left| \det \left[\frac{\partial \zeta_{\alpha,i}}{\partial x_j} \left(\varphi_\alpha^{-1}(\boldsymbol{y}); \varphi_0(e) \right) \right]_{i,j} \right|_p^{-1} \mathrm{d}\lambda(\boldsymbol{y}), \qquad (1)$$

where (U_0, φ_0) is a chart around $e \in G$, with local coordinates $(x_j)_j$, $\zeta_{\alpha,i}(\varphi_\alpha^{-1}(\boldsymbol{y}); \boldsymbol{x}) := \varphi_{\alpha,i}(\varphi_\alpha^{-1}(\boldsymbol{y})\varphi_0^{-1}(\boldsymbol{x}))$, and λ is the Haar measure on $\mathbb{Q}_p^n$.

[3] Aniello, L'Innocente, Mancini, Parisi, Svampa, Winter, "Invariant measures on p-adic Lie groups: the p-adic quaternion algebra and the Haar integral on the p-adic rotation groups", *Lett. Math. Phys.* **114**(78) (2024).

Haar integral on $\mathrm{SO}(3)_p$

The p-adic quaternion algebra $\mathbb{H}_p$ is the division algebra over $\mathbb{Q}_p$ with basis $(1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} := \boldsymbol{ij})$ satisfying $\boldsymbol{i}^2 = v,\ \boldsymbol{j}^2 = -p,\ \boldsymbol{ji} = -\boldsymbol{ij}$.

$\mathbb{H}_p \ni \xi = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$

- Haar measure on $\mathbb{H}_p^\times$: $\quad \mu_{\mathbb{H}_p^\times}(E) = \int_{\varphi(E)} \frac{\mathrm{d}\lambda(\boldsymbol{q})}{|Q_+^{(4)}(\boldsymbol{q})|_p^2}$
  where $\varphi(\xi) := \boldsymbol{q} = (q_0, q_1, q_2, q_3)$
- Isomorphism $\quad \mathrm{SO}(3)_p \simeq \mathbb{H}_p^\times / \mathbb{Q}_p^\times$
- Weil-Mackey-Bruhat lift

$$\int_{\mathrm{SO}(3)_p} \mathrm{d}\mu(\mathcal{R})\phi(\mathcal{R}) = \int_{\mathbb{H}_p^\times} \mathrm{d}\mu_{\mathbb{H}_p^\times}(\boldsymbol{q})(\mathscr{L}\phi)(\boldsymbol{q})$$

2^{nd} approach

$\mathrm{SO}(3)_p \mod p^k$

- $\mathrm{SO}(3)_p \subset \mathrm{M}(3, \mathbb{Z}_p)$ is a profinite group
- Homomorphism for $k \in \mathbb{N}$:

$$\pi_k : \mathrm{SO}(3)_p \to \pi_k\left(\mathrm{SO}(3)_p\right) \subset \mathrm{M}(3, \mathbb{Z}/p^k\mathbb{Z})$$
$$\pi_k(\mathcal{R}) = \mathcal{R} \mod p^k \quad \text{entry-wise}$$

- Finite groups $G_{p^k} := \pi_k\left(\mathrm{SO}(3)_p\right)$

$$\ldots \twoheadrightarrow G_{p^{k+1}} \overset{\mathrm{mod}\ p^k}{\twoheadrightarrow} G_{p^k} \twoheadrightarrow \ldots \twoheadrightarrow G_{p^2} \overset{\mathrm{mod}\ p}{\twoheadrightarrow} G_p$$

- $\mathrm{SO}(3)_p = \varprojlim \{G_{p^k}\}_{\mathbb{N}}$
- Nautical parameters and multivariable Hensel's lift

$$|G_{p^k}| = 2p^{3k-1}(p+1)$$

Inverse-limit Haar measure on $\mathrm{SO}(3)_p$ [4]

Proposition

Let μ_k be the Haar measure on G_{p^k}, with μ_k, μ_l coherent for $k < l$. There exists a unique measure μ on $\mathrm{SO}(3)_p$ coherent with each μ_k, and it is the Haar measure on $\mathrm{SO}(3)_p$.

- Power set Σ_k of G_{p^k}
  Normalised counting measure $\mu_k(E_k) := \frac{|E_k|}{|G_{p^k}|}$
- $\Sigma_k^\star := \pi_k^{-1}(\Sigma_k)$ on $\mathrm{SO}(3)_p$, with measure $\mu_k^\star(E) := (\mu_k \circ \pi_k)(E)$
- $A := \bigcup_{k\in\mathbb{N}} \Sigma_k^\star$ not a σ-algebra,
  $\widetilde{\mu}(E) := \mu_k^\star(E)$ for $E \in \Sigma_k^\star$, σ-additive
- $\Sigma(A) = \mathcal{B}_{\mathrm{SO}(3)_p}$ with unique σ-additive extension of $\widetilde{\mu}$:

$$\mu(E) := \inf \left\{\widetilde{\mu}(F) \text{ s.t. } E \subseteq F \in A\right\} = \inf_k \left\{\widetilde{\mu}\left(\pi_k^{-1}(\pi_k(E))\right)\right\}$$

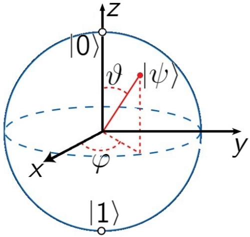
[4] Aniello, L'Innocente, Mancini, Parisi, Svampa, Winter, "Characterising the Haar measure on the p-adic rotation groups via inverse limits of measure spaces", *Expo. Math.* **43**(2) (2025).

Representations of $\mathrm{SO}(3)_p$

Euclidean scenario

$$\mathrm{SO}(3)_{\mathbb{R}} \simeq \mathrm{SU}(2)/\{\pm \mathrm{I}\}$$

- $(2l+1)$-dim irreps indexed by integer or half-integer l
- The state space of the qubit is $\mathbb{C}^2$
  Vector states on the Bloch sphere



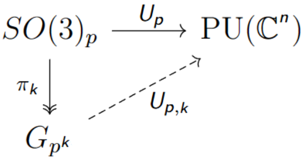
- Action of $\mathrm{SU}(2)$ on $\mathbb{C}^2$ corresponds to projective action of $\mathrm{SO}(3)_{\mathbb{R}}$ on the Bloch sphere
- *p*-Adic qubit as a projective irrep of $\mathrm{SO}(3)_p$ on $\mathbb{C}^2$

Factorisation of irreps

- Irrep $U_{p,k}$ of G_{p^k} $\rightsquigarrow$ irrep U_p of $\mathrm{SO}(3)_p$
$$U_p := U_{p,k} \circ \pi_k$$

Proposition

For every irrep U_p of $\mathrm{SO}(3)_p$, there exists $k \in \mathbb{N}$, irrep $U_{p,k}$ of G_{p^k}, such that $U_p = U_{p,k} \circ \pi_k$.



- Exhaustive method
- Start from $k = 1$

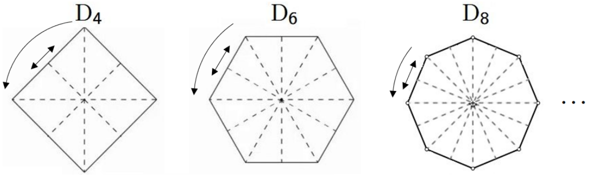
From rotations to dihedral groups [5]

- $G_p \ni M(a,b,c,d,s) = \begin{pmatrix} a & svb & 0 \\ b & sa & 0 \\ c & d & s \end{pmatrix}$, $a^2 - vb^2 \equiv 1,\ s \equiv \pm 1 \mod p$
- Homomorphism $\mathcal{F}_p \colon G_p \twoheadrightarrow \mathcal{F}_p(G_p)$, $\begin{pmatrix} a & svb & 0 \\ b & sa & 0 \\ c & d & s \end{pmatrix} \mapsto \begin{pmatrix} a & svb \\ b & sa \end{pmatrix}$
- $\mathcal{F}_p(G_p) = \Big\langle C := \begin{pmatrix} a_0 & vb_0 \\ b_0 & a_0 \end{pmatrix},\ Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ s.t. $C^{p+1} \equiv Z^2 \equiv \mathrm{I}_2, ZCZ \equiv C^{-1} \Big\rangle$
  $\simeq \mathrm{D}_{p+1}$

Onto homomorphism $\phi_p \colon G_p \to \mathrm{D}_{p+1}$

[5] Svampa, Mancini, Winter, "An approach to p-adic qubits from irreducible representations of $\mathrm{SO}(3)_p$", *J. Math. Phys.* **63**(7), 2022.

Some p-adic qubit representations



D_{p+1} has:

four 1-dim irreps

$\frac{p-1}{2}$ 2-dim irreps $\sigma_p^{(i)}$

$\Downarrow$

many p-adic qubits for $p > 3$

p-Adic qubit representations:

$$U_p^{(i)} : \mathrm{SO}(3)_p \to \mathrm{U}(2), \qquad U_p^{(i)} := \sigma_p^{(i)} \circ \phi_p \circ \pi_1$$

$p = 3$ explicitly: four other 4-dim irreps

p-Adic quantum computing [6]

- The p-adic qubit is the fundamental object
- Its pure states are on $\mathbb{C}^2$
- Composite systems of qubits via tensor-product representations
- Clebsch-Gordan decomposition

Standard QM
Unique 2-dim irrep U
$U^{\otimes n} \rightsquigarrow$ all irreps

p-Adic QM
More 2-dim irreps
$U_p = U_{p,k} \circ \pi_k \Rightarrow U_p^{\otimes n} = U_{p,k}^{\otimes n} \circ \pi_k$

[6] L'Innocente, Mancini, Svampa, Winter, "Putting together p-adic qubits: from representations of $\mathrm{SO}(3)_p$ to entanglement and logic gates", in preparation.

p-Adic quantum computing [6]

- Clebsch-Gordan decomposition

Standard QM

$2 \otimes 2 \simeq 1 \oplus 3$

p-Adic QM

For the 2-dim irreps of D_{p+1}:

$2 \otimes 2 \simeq 1 \oplus 1 \oplus 1 \oplus 1$

$2 \otimes 2 \simeq 1 \oplus 1 \oplus 2$

$2 \otimes 2 \simeq 2 \oplus 2$

- Entanglement

Singlet $|\Psi^-\rangle$

Triplet $(|00\rangle, |\Psi^+\rangle, |11\rangle)$

Singlets or doublets

$|\Phi^\pm\rangle, |\Psi^\pm\rangle$

where $|\Phi^\pm\rangle := \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\Psi^\pm\rangle := \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are the four maximally entangled Bell states

[6] L'Innocente, Mancini, Svampa, Winter, "Putting together p-adic qubits: from representations of $\mathrm{SO}(3)_p$ to entanglement and logic gates", in preparation.

p-Adically controlled logic gates

- Entangling two-qubit gate?
- GAP for G_3
- Character table

G_3	I	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
χ_{TRIV}	1	1	1	1	1	1	1	1	1
$\chi_{1,1D}$	1	1	1	1	1	-1	-1	-1	-1
$\chi_{2,1D}$	1	1	1	1	-1	1	1	-1	-1
$\chi_{3,1D}$	1	1	1	1	-1	-1	-1	1	1
χ_{2D}	2	2	2	-2	0	0	0	0	0
$\chi_{1,4D}$	4	1	-2	0	0	-2	1	0	0
$\chi_{2,4D}$	4	1	-2	0	0	2	-1	0	0
$\chi_{3,4D}$	4	-2	1	0	0	0	0	-2	1
$\chi_{4,4D}$	4	-2	1	0	0	0	0	2	-1

- Explicit 4-dim irreps

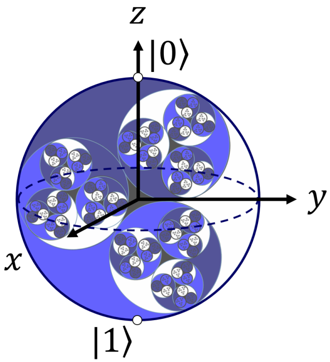
Conclusions

- There is a unique group $\mathrm{SO}(3)_p$.
  We studied its geometric and topological properties: rotations, nautical decompositions, inverse limit, quaternions, Hensel's lift.
- We expressed the Haar measure on $\mathrm{SO}(3)_p$ through two different approaches: integral and inverse limit.
- We studied the irreps of $\mathrm{SO}(3)_p$.
  They all factorise modulo p^k for some k.
  We found explicit p-adic qubit representations for every prime p.
- We laid the foundations of a p-adic theory of angular momentum and spin, as well as of p-adically controlled quantum computation

Open questions

- $p = 2$ is always peculiar
- Structure of $\mathrm{SO}(2)_{p,\ell}$
- Haar measure in terms of nautical parameters
- Number of 1-dim irreps
  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \subseteq \mathrm{SO}(3)_p/[\mathrm{SO}(3)_p, \mathrm{SO}(3)_p]$
  Number of 2-dim irreps $\leftrightarrow$ p-adic qubits
- Irreps of $\mathrm{SO}(3)_p$ mod p^k
- Universal set of p-adically controlled gates

Thank You!



Parametrisation of $\mathrm{SO}(2)_{p,\ell}$

$$\mathrm{SO}(2)_{p,\ell} \ni \mathcal{R}_\ell(\sigma) = \begin{pmatrix} \frac{1-\ell\sigma^2}{1+\ell\sigma^2} & -\frac{2\ell\sigma}{1+\ell\sigma^2} \\ \frac{2\sigma}{1+\ell\sigma^2} & \frac{1-\ell\sigma^2}{1+\ell\sigma^2} \end{pmatrix}, \qquad \sigma \in \mathbb{Q}_p \cup \{\infty\}$$

$$\mathcal{R}_\ell\left(-\frac{1}{\ell\sigma}\right) = -\mathcal{R}_\ell\left(\sigma\right)$$

Either $\sigma \in \mathbb{Z}_p$ or $\sigma = -\frac{1}{\ell\tau}$ with

- $\tau \in p\mathbb{Z}_p$ for $\ell = -v$:

$$\mathrm{SO}(2)_{p,-v} = \{\mathcal{R}_{-v}(\sigma) \text{ s.t. } \sigma \in \mathbb{Z}_p\} \cup \{-\mathcal{R}_{-v}\left(\sigma\right) \text{ s.t. } \sigma \in p\mathbb{Z}_p\}$$

- $\tau \in \mathbb{Z}_p$ for $\ell \in \{p, up\}$:

$$\mathrm{SO}(2)_{p,\ell} = \{\pm\mathcal{R}_\ell(\sigma) \text{ s.t. } \sigma \in \mathbb{Z}_p\}$$

Proof of main-angle decompositions

$\mathcal{R} = \mathcal{R}_{\boldsymbol{n}_1}(\sigma)\mathcal{R}_{\boldsymbol{n}_2}(\tau)\mathcal{R}_{\boldsymbol{n}_3}(\omega)$ iff $\mathcal{R}_{\boldsymbol{n}_2}(\tau)^{-1}\mathcal{R}_{\boldsymbol{n}_1}(\sigma)^{-1}\mathcal{R}\boldsymbol{n}_3 = \boldsymbol{n}_3$
iff there exists $\mathcal{R}_{\boldsymbol{n}_1}(\sigma) \in \mathrm{SO}(3)_p$ s.t.

$$\mathcal{R}_{\boldsymbol{n}_1}(\sigma)^{-1}\mathcal{R}\boldsymbol{n}_3 \perp \boldsymbol{n}_2 \tag{2}$$

$$Q_+\left(\mathcal{R}_{\boldsymbol{n}_1}(\sigma)^{-1}\mathcal{R}\boldsymbol{n}_3\right) = Q_+(\boldsymbol{n}_3) \tag{3}$$

- E.g. $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3) = (\boldsymbol{e}_3, \boldsymbol{e}_2, \boldsymbol{e}_1)$
  Is there $x_1 \in \mathbb{Q}_p$ such that $x_1^2 + px_2^2 = 1$ where $x_2 \in \mathbb{Z}_p$?
  By Hensel's Lemma
- Non-existence with counterexamples on Eqs. (2), (3)
- Duplicity due to $\mathcal{R}_x(\infty)\mathcal{R}_y(\infty)\mathcal{R}_z(\infty) = \mathrm{I}_3$ and solutions of one quadratic equation in one unknown

Construction of Haar measure on *p*-adic Lie groups

- *p*-adic Lie group G, disjoint atlas $\mathcal{A} = \{U_\alpha, \varphi_\alpha\}_\alpha$
- Radon measure ν on G by $\nu_\alpha := {\varphi_\alpha^{-1}}_* \left(\lambda_{|\varphi_\alpha(U_\alpha)}\right)$
- $\nu^h(E) := \nu(hE)$, for a Borel set E and $h \in G$
- ν quasi-left-invariant $\Rightarrow$ $\mathrm{d}\nu^h(g) = \eta(h,g)\mathrm{d}\nu(g)$ with

$$\eta(h,g) = \left| \det \left(\frac{\partial \zeta_{\beta,i}}{\partial x_j}(h; \boldsymbol{x} = \varphi_\alpha(g)) \right)_{i,j} \right|_p ,$$

$$\zeta_{\beta,i}(h; \boldsymbol{x}) := \varphi_{\beta,i}(h\varphi_\alpha^{-1}(\boldsymbol{x}))$$

- left Haar measure

$$\mathrm{d}\mu(g) := \eta(g,e)^{-1}\mathrm{d}\nu(g)$$

Haar integrals on $\mathrm{SO}(2)_{p,\ell}$, $\mathrm{SO}(4)_p$

- On $\mathrm{SO}(2)_{p,\ell}$, with $\varphi_\ell^{-1}(\sigma) := \mathcal{R}_\ell(\sigma)$,

$$\mu(E) = \int_{\varphi_\ell(E)} \frac{1}{|1+\ell\sigma^2|_p} \mathrm{d}\lambda(\sigma)$$

- $\mathrm{SO(4)_p} \simeq \mathbb{P}(\mathbb{H}_p^\times)/\mathbb{Q}_p^\times$, where $\mathbb{P}(\mathbb{H}_p^\times) := \{(\xi,\rho) \in \mathbb{H}_p^\times \times \mathbb{H}_p^\times \text{ s.t. } \mathrm{nrd}(\xi) = \mathrm{nrd}(\rho)\}$
- Haar measure on a neighbourhood of the identity of $\mathbb{P}(\mathbb{H}_p^\times)$
- Haar integral on a neighbourhood of the identity of $\mathrm{SO}(4)_p$ via Weil-Mackey-Bruhat lift

Multivariable Hensel lift for $\mathrm{SO}(3)_p$

$$\widetilde{G}_{p^k} := \left\{ \widetilde{L} \in \mathrm{M}\left(3, \mathbb{Z}/p^k\mathbb{Z}\right) \text{ s.t. } \widetilde{L}^\top \pi_k\left(A_+\right)\widetilde{L} = \pi_k\left(A_+\right), \ \det \widetilde{L} \equiv 1 \right\}$$

$$G_{p^k} \subseteq \widetilde{G}_{p^k}$$

Is the converse true?

Theorem

If $L = (\ell_{ij})_{i,j} \in \mathrm{M}(3, \mathbb{Z}_p)$ is a solution modulo p^k, there exists $Z = (z_{ij})_{i,j} \in \mathrm{M}(3, \mathbb{Z}_p)$ such that $L + p^k Z$ is solution modulo p^{k+1}, for every $k \in \mathbb{N}$.
Any solution L modulo p^k admits exactly p^3 distinct lifted solutions $L + p^k Z$ modulo p^{k+1}.

Proof of factorisation of irreps modulo p^k

- G profinite, representation $\rho : G \to \mathrm{GL}(n, \mathbb{C})$
- ρ continuous iff $\ker(\rho)$ open
  no small subgroups for $\mathrm{GL}(n, \mathbb{C})$
- factorisation of ρ,
  $\ker(\pi_k) \subseteq \ker(\rho)$
- extend to projective representations $\underline{U} : G \to \mathrm{PU}(n)$
  $\underline{U}$ induces $\rho : G \to \mathrm{U}\big(\mathrm{M}(n, \mathbb{C})\big)$, $\rho(g)M := U(g)MU(g)^\dagger$
  $\ker(\rho) = \ker(\underline{U})$